Practice problems 1

Material from Part 1

Due at the beginning of recitation on Friday, September 4, 2015.

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Problem 1: De Morgan’s laws.
Suppose that $A$ lies in a larger set $X$, $A \subseteq X$. Define the complement of $A$ with respect to $X$ as

$$A^C = X \setminus A.$$  

Prove De Morgan’s laws:

i. $A^C \cap B^C = (A \cup B)^C$.

ii. $A^C \cup B^C = (A \cap B)^C$.

Problem 2: Associative operators.
Prove the following equalities:

i. $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$.

ii. $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$.

Problem 3: Series of injections.
Suppose that there is an injection $f : X \to Y$ and an injection $g : Y \to Z$.
Prove that there is an injection $h : X \to Z$.

Problem 4: Bijection to a proper subset.
Suppose that $B \subset A$, and $B$ is finite.

i. Show that there is no bijection $f : A \to B$. Hint: separate the cases where $A$ is finite and $A$ is infinite.

ii. Why may part i. not hold if $B$ is infinite? Hint: if $B$ is infinite and $B \subset A$, $A$ must be infinite.

Problem 5: Lots of numbers.
Prove that $\mathbb{N}^n$ is countable for all $n \in \mathbb{N}$. Hint: you have seen that $\mathbb{N} \times \mathbb{N} = \mathbb{N}^2$ is countable, so there is a bijection $h : \mathbb{N} \to \mathbb{N}^2$. Use an argument by induction to obtain the desired result.
Problem 6: Size of an interval.
Prove that for any $a, b \in \mathbb{R}$, $a < b$, it is the case that $|[a, b]| = |\mathbb{R}|$. **Hint:** one direction is straightforward. For the other, try proving $|[-1, 1]| = |\mathbb{R}|$ and either adapt your argument to $[a, b]$, or remember facts about compositions of bijections.