Problem set 2

Due at the beginning of recitation on Friday, September 18, 2015.

Problem 1: Constant subsequences.
Let \( \langle x_k \rangle_{k=0}^{\infty} \) be a sequence in \( \mathbb{R} \), and define a correspondence \( I : \mathbb{R} \rightarrow \mathbb{N} \) as
\[
I(x) = \{ k \in \mathbb{N} : x_k = x \}.
\]
Show that if there are \( x, x' \in \mathbb{R}, x \neq x' \) such that \( I(x) \) and \( I(x') \) are infinite, then \( \langle x_k \rangle_{k=0}^{\infty} \) does not converge.

Problem 2: Convergent differences.
Let \( \langle x_k \rangle_{k=0}^{\infty} \) and \( \langle y_k \rangle_{k=0}^{\infty} \) be sequences in \( \mathbb{R} \), where \( x_k \rightarrow x^* \) and \( y_k \rightarrow y^* \).

i. Prove that \( \langle x_k - y_k \rangle_{k=0}^{\infty} \rightarrow x^* - y^* \).

ii. Let \( x^* = y^* = c \). Show that any sequence \( \langle z_k \rangle_{k=0}^{\infty} \) with \( \min \{ x_k, y_k \} \leq z_k \leq \max \{ x_k, y_k \} \) for all \( k \in \mathbb{N} \) is such that \( z_k \rightarrow c \).

Problem 3: Problems with rationality.

i. Show by example that a countable intersection of dense open sets may not be open.

ii. Show that \([0,1] \cap \mathbb{Q} \) is not compact.

Problem 4: Sequential open intervals.
Suppose that \( \langle a_k \rangle_{k=0}^{\infty} \) and \( \langle b_k \rangle_{k=0}^{\infty} \) are monotone convergent sequences in \( \mathbb{R} \), \( a_k < b_k \) for all \( k \), and there is \( c \in \mathbb{R} \) such that \( a_k \rightarrow c \) and \( b_k \rightarrow c \). Show that if \( \cap_{k=0}^{\infty} (a_k, b_k) = \{ c \} \), then \( \langle a_k \rangle_{k=0}^{\infty} \) has a strictly increasing subsequence and \( \langle b_k \rangle_{k=0}^{\infty} \) has a strictly decreasing subsequence.

Problem 5: Cauchy subsequences.
Let \( \langle x_k \rangle_{k=0}^{\infty} \) be a Cauchy sequence.

i. For any \( k, j \in \mathbb{N} \), let \( y_{k,j} = x_{k+j} - x_k \). Show that there is \( y_k^* \) such that \( \langle y_{k,j} \rangle_{j=0}^{\infty} \rightarrow y_k^* \).

ii. Show that \( \langle y_k^* \rangle_{k=0}^{\infty} \) converges to 0.

iii. Let \( f : \mathbb{N} \rightarrow \{-1, 1\} \), and let \( \langle z_k \rangle_{k=0}^{\infty} \) be such that \( z_k = \sum_{j=0}^{k} f(j)x_j \). Show that \( \langle z_k \rangle_{k=0}^{\infty} \) converges only if \( x_k \rightarrow 0 \).

Problem 6: Monotone functions.
Let \( X \subseteq \mathbb{R} \) be compact and \( f : \mathbb{R} \rightarrow \mathbb{R} \) be monotone. Show that there is \( x \in X \) such that \( f(x) = \sup \{ f(x') : x' \in X \} \).