Problem set 3

Material from Part 3

Due at the beginning of recitation on Friday, October 16, 2015.

Problem 1: Transformation of distance.
Let \((X, d)\) be a metric space and \(f : \mathbb{R}^+ \to \mathbb{R}^+\) be monotone strictly increasing. Must \((X, f \circ d)\) be a metric space?

Problem 2: Naturals.
Let \((\mathbb{N}^+, d_i)\) be a metric space.

i. Let \(d_1(n_1, n_2) = |n_1 - n_2|\). Confirm that \((\mathbb{N}^+, d_1)\) is a metric space, and show that it is unbounded.

ii. Let \(d_2(n_1, n_2) = |1/n_1 - 1/n_2|\). Confirm that \((\mathbb{N}^+, d_2)\) is a metric space, and show that it is bounded.

Problem 3: Jacard distance.
Let \(M\) be a finite set, and \(X = \mathcal{P}(M) \setminus \{\emptyset\}\). Let \(d : X^2 \to \mathbb{R}\) be
\[
d(Y, Z) = \frac{|Y \cup Z| - |Y \cap Z|}{|Y \cup Z|}
\]
Show that \((X, d)\) is a metric space. **Hint:** with regard to the triangle inequality, which \(C\) maximizes \(d(A, C) + d(B, C)\)?

Problem 4: Connectedness.
A set \(X\) is connected if there are no two nonempty open sets \(Y_1, Y_2\) such that \(Y_1 \cup Y_2 \supseteq X\) and \(Y_1 \cap Y_2 = \emptyset\). Let \((\mathbb{R}, d)\) be the standard Euclidean metric space on the reals. Show that \(Z \subseteq \mathbb{R}\) is not connected if \((\inf Z, \sup Z) \not\subseteq Z\).

Problem 5: Continuity.
Let \(X = [0, 1/3] \cup [2/3, 1]\), and \(d(x, y) = |x - y|\). Show that \(f : X \to X, f(x) = x\) is continuous.

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1Under mild conditions this can be extended to “if and only if,” but the proof is more difficult.
Problem 6: Continuity of distance.
Let \((X, d_X), (X^2, d_{X^2}), \) and \((\mathbb{R}, d_{\mathbb{R}})\) be metric spaces, where \(d_{\mathbb{R}}(x, y) = |x - y|\) and \(d_{X^2}(x, y) = d_X(x_1, y_1) + d_X(x_2, y_2)\). Show that \(d_X : X^2 \to \mathbb{R}\) is continuous.

Problem 7: Compactness.
Let \((\mathbb{R}^{++}, d)\) be a metric space,
\[
d(x, y) = \begin{cases} 
0 & \text{if } x = y, \\
|x| + |y| & \text{otherwise.}
\end{cases}
\]
Show that \(Y \subseteq \mathbb{R}\) is compact if and only if it is finite. What breaks if we consider \(\mathbb{R}^+ = \mathbb{R}^{++} \cup \{0\}\)?

Problem 8: Functions.
Let \(X = \{f : f \text{ is continuous and } f : [0, 1] \to [0, 1]\}\) and \(d : X^2 \to \mathbb{R}\) be
\[
d(f, g) = \sup \{|f(x) - g(x)| : x \in [0, 1]\}.
\]
Let \(\langle f_k \rangle_{k=1}^\infty\) be a sequence in \(X\), \(f_k(x) = x^{1/k}\) for all \(k, x\).

i. Prove that \(\langle f_k \rangle_{k=1}^\infty\) does not converge in \(X\).

ii. Prove that \(\langle f_k \rangle_{k=1}^\infty\) is not Cauchy.

Problem 9: Topology.
For each \(i \in \{1, 2\}\) let \(\emptyset \subset Y_i \subset \mathbb{R}\), and let \(T_i = (\mathbb{R}, \{\emptyset, Y_i, \mathbb{R}\})\) be a topological space.

i. Confirm that \(T_i\) is a topological space.

ii. Let \(f : \mathbb{R} \to \mathbb{R}\), \(f(x) = x\) for all \(x \in \mathbb{R}\). Show that \(f\) is continuous (as a map from \(T_1\) to \(T_2\)) if and only if \(Y_1 = Y_2\).

Problem 10: Topology.
Let \(X\) be a set and \(\mathcal{F} = \mathcal{P}(X)\) be the set of subsets of \(X\).

i. Show that \((X, \mathcal{F})\) is a topological space.

ii. Let \(\mathcal{F}' = \{X' \in \mathcal{P}(X) : |X'| < |X| - 1\} \cup \{X\}\). Show that \((X, \mathcal{F}')\) is not a topological space. Hint 1: \(X = \cup_{x \in X} \{x\}\). Hint 2: if \(X\) is infinite, \(|X| = |X| - 1\).