1 Games in Normal Form (Strategic Form)

A Game in Normal (strategic) Form consists of three components:

1. A set of players

2. For each player, a set of strategies (called actions in textbook). The interpretation is that this should be thought of as anything the player can do.

3. For each player, some preferences over the set of strategy profiles.

We will let $N$ denote the set of agents, which is assumed to be a finite set. For each player $i \in N$ we denote by $S_i$ the strategy set. To reduce notational clutter we let $S = \times_{i=1}^n S_i$ (the Cartesian product), which allows us to write a payoff function for player $i$ as $u_i : S \rightarrow R$.

**Definition 1** A normal form game is an triple $G = (N, S, u)$ where $N = \{1,..,n\}$ is the set of players, $S = \times_{i=1}^n S_i$ is the strategy space and $u = (u_1,\ldots,u_n)$ are the payoff functions for player.

1.1 Language/Notation

It is convenient to establish some language and shorthand notation for future use:

1. We write $s_i$ for a generic element in $S_i$ and refer to it as a strategy.

2. We write $s = (s_1,\ldots,s_n)$ for a vector consisting of a strategy for each player. We refer to this as a strategy profile.

3. We write $s_{-i} = (s_1,\ldots,s_{i-1},s_{i+1},\ldots,s_n)$ for a strategy profile where the $i$th coordinate has been removed.

4. We will also use $S_{-i}$ for $\times_{j\neq i} S_j$, so that $S_{-i}$ consists of every conceivable $s_{-i}$ that the players other than $i$ may pick.
5. The latter convention allows us to write \((s'_i, s_{-i})\) for the strategy profile \((s_1, \ldots, s_{i-1}, s'_i, s_{i+1}, \ldots, s_n)\), which will be convenient as equilibrium concepts are defined by having each player \(i\) optimize given \(s_{-i}\).

### 1.2 Dominant and Dominated Strategies

**Definition 2** \(s'_i\) is strictly dominated by \(s''_i\) if \(u_i(s'_i, s_{-i}) < u_i(s''_i, s_{-i})\) for every \(s_{-i} \in S_{-i}\).

**Definition 3** \(s'_i\) is weakly dominated by \(s''_i\) if \(u_i(s'_i, s_{-i}) \leq u_i(s''_i, s_{-i})\) for every \(s_{-i} \in S_{-i}\) and if \(u_i(s'_i, s_{-i}) < u_i(s''_i, s_{-i})\) for some \(s_{-i} \in S_{-i}\).

**Remark 1** This is nonstandard in that the typical notion of dominance allows for randomizations. We will return to this when talking about randomized strategies.

**Remark 2** \(s'_i\) is strictly (weakly) dominated by \(s''_i\) or \(s''_i\) strictly (weakly) dominates \(s'_i\) are used interchangeably.

**Definition 4** \(s'_i\) is a strictly dominant strategy if \(u_i(s'_i, s_{-i}) > u_i(s_i, s_{-i})\) for every \(s_{-i} \in S_{-i}\) and every \(s_i \in S_i \setminus \{s'_i\}\).

**Definition 5** \(s'_i\) is a weakly dominant strategy if \(u_i(s'_i, s_{-i}) \geq u_i(s_i, s_{-i})\) for every \(s_{-i} \in S_{-i}\) and every \(s_i \in S_i \setminus \{s'_i\}\) and if for every \(s_i \in S_i \setminus \{s'_i\}\) \(u_i(s'_i, s_{-i}) > u_i(s_i, s_{-i})\) for some \(s_{-i} \in S_{-i}\).

### 1.2.1 The Prisoners Dilemma

Suppose that \(N = \{1, 2\}\), \(S_1 = S_2 = \{\text{Defect, Cooperate}\}\) and that

\[
\begin{align*}
u_1 (\text{Defect, Cooperate}) &> u_1 (\text{Cooperate, Cooperate}) > u_1 (\text{Defect, Defect}) > u_1 (\text{Cooperate, Defect}) \\
u_2 (\text{Cooperate, Defect}) &> u_2 (\text{Cooperate, Cooperate}) > u_2 (\text{Defect, Defect}) > u_2 (\text{Defect, Cooperate})
\end{align*}
\]
Let

\[ u_1 (\text{Defect, Cooperate}) = u_2 (\text{Cooperate, Defect}) = a \]
\[ u_1 (\text{Cooperate, Cooperate}) = u_2 (\text{Cooperate, Cooperate}) = b \]
\[ u_1 (\text{Defect, Defect}) = u_2 (\text{Defect, Defect}) = c \]
\[ u_1 (\text{Cooperate, Defect}) = u_2 (\text{Defect, Cooperate}) = d; \]

where \( a > b > c > d \) in order to capture the prisoner’s dilemma ranking. It is convenient to collect this information in a payoff bi-matrix

<table>
<thead>
<tr>
<th>Player 2</th>
<th>Defect</th>
<th>Cooperate</th>
</tr>
</thead>
<tbody>
<tr>
<td>Player 1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Defect</td>
<td>( c, c )</td>
<td>( a, d )</td>
</tr>
<tr>
<td>Cooperate</td>
<td>( d, a )</td>
<td>( b, b )</td>
</tr>
</tbody>
</table>

and since \( c > d \) and \( a > b \) it is a strictly dominant to play defect. Hence, \((\text{Defect, Defect})\) is an equilibrium in dominant strategies.

### 1.2.2 An Alternative Story

Suppose Alice and Bob have a lawn moving operation. They can either work (hard) or shirk, and the total profit is 10 if both work hard 6 if one of them works hard and 2 if both shirks. Also assume that the cost of effort is \( e \) and that they split the profit equally. This gives the following payoff matrix

<table>
<thead>
<tr>
<th>Bob</th>
<th>Shirk</th>
<th>Work hard</th>
</tr>
</thead>
<tbody>
<tr>
<td>Alice</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Shirk</td>
<td>( 1, 1 )</td>
<td>( 3, 3 - e )</td>
</tr>
<tr>
<td>Work hard</td>
<td>( 3 - e, 3 )</td>
<td>( 5 - e, 5 - e )</td>
</tr>
</tbody>
</table>
Hence, with cost of effort $e = 3$ we get

<table>
<thead>
<tr>
<th></th>
<th>Bob</th>
<th>Shirk</th>
<th>Work hard</th>
</tr>
</thead>
<tbody>
<tr>
<td>Alice</td>
<td>Shirk</td>
<td>1, 1</td>
<td>3, 0</td>
</tr>
<tr>
<td></td>
<td>Work hard</td>
<td>0, 3</td>
<td>2, 2</td>
</tr>
</tbody>
</table>

which is identical a Prisoner’s dilemma.

### 1.2.3 A Second Price Auction with 2 Bidders (Vickrey Auction)

Let $N = \{1, 2\}$, $S_1 = S_2 = R_+$ and

$$u_1 (b_1, b_2) = \begin{cases} v_1 - b_2 & \text{if } b_1 > b_2 \\ \frac{v_1 - b_2}{2} & \text{if } b_1 = b_2 \\ 0 & \text{if } b_1 < b_2 \end{cases}$$

$$u_2 (b_1, b_2) = \begin{cases} v_2 - b_1 & \text{if } b_1 < b_2 \\ \frac{v_2 - b_1}{2} & \text{if } b_1 = b_2 \\ 0 & \text{if } b_1 > b_2 \end{cases}$$

Clearly, there is no strictly dominant strategy as, for example,

$$u_1 (b_1', b_2) = u_1 (b_1'', b_2) = v_1 - b_2$$

for every $(b_1', b_1'')$ such that $b_1' > b_1'' > b_2$.

**Proposition 1** *The unique weakly dominant strategy is to bid $b_i = v_i$.***

**Proof.** Without loss, consider $i = 1$. Suppose that $v_1 > b_2$ then

$$u_1 (v_1, b_2) = v_1 - b_2 > 0$$

$$u_1 (b_1, b_2) = \begin{cases} v_1 - b_2 = u_1 (v_1, b_2) & \text{if } b_1 > b_2 \\ \frac{v_1 - b_2}{2} = \frac{u_1 (v_1, b_2)}{2} < u_1 (v_1, b_2) & \text{if } b_1 = b_2 \\ 0 < u_1 (v_1, b_2) & \text{if } b_1 < b_2 \end{cases}$$
Suppose that $v_1 = b_2$ then

\[
u_1 (v_1, b_2) = v_1 - b_2 = 0
\]
\[
u_1 (b_1, b_2) = \begin{cases} v_1 - b_2 = 0 & \text{if } b_1 > v_1 = b_2 \\ 0 & \text{if } b_1 < v_1 = b_2 \end{cases}
\]

Suppose that $v_1 < b_2$ then

\[
u_1 (v_1, b_2) = v_1 - b_2 = 0
\]
\[
u_1 (b_1, b_2) = \begin{cases} v_1 - b_2 < 0 & \text{if } b_1 > b_2 \\ \frac{v_1 - b_2}{2} < 0 & \text{if } b_1 = b_2 \\ 0 & \text{if } b_1 < b_2 \end{cases}
\]

Hence, in each case bidding the valuation is optimal, so it is weakly dominant.

Next, generalize to the case of $N = \{1, \ldots, n\}$

\[
u_1 (b_1, \ldots, b_n) = \begin{cases} v_i - \max_{j \neq i} b_j & \text{if } b_i > \max_{j \neq i} b_j \\ \frac{v_i - \max_{j \neq i} b_j}{|\arg \max_{j \in N} b_j|} & \text{if } b_i = \max_{j \neq i} b_j \\ 0 & \text{if } b_i < \max_{j \neq i} b_j \end{cases}
\]

The proof is identical once we replace $b_2$ with $\max_{j \neq i} b_j$, so again it is a weakly dominant strategy to bid the valuation.

**1.2.4 The Groves Mechanism**

Suppose that $N = \{1, \ldots, n\}$, that $x \in X$ is some social decision with cost function $C(x)$ and that preferences are given by

\[u_i (x, v_i) - t\]

where $v_i$ is a preference parameter that is also an element of some set $V_i$. To generate a normal form assume that each player $i$ may choose an $\hat{v}_i \in V_i$ and that given strategy profile
\( \hat{v} = (\hat{v}_1, \ldots, \hat{v}_n) \) a social decision

\[
x^* (\hat{v}) \in \arg \max_{x \in X} \sum_{i=1}^{n} u_i (x, \hat{v}_i) - C(x)
\]

\[
t_i (\hat{v}) = C(x^* (\hat{v})) - \sum_{j \neq i} u_j (x^* (\hat{v}), \hat{v}_j).
\]

Hence, we have a normal form game \((N, V, \pi)\) where the payoff \(\pi_i\) is given by

\[
\pi_i (\hat{v}) = u_i (x^* (\hat{v}), v_i) - t_i (\hat{v})
\]

\[
= u_i (x^* (\hat{v}), v_i) + \sum_{j \neq i} u_j (x^* (\hat{v}), \hat{v}_j) - C(x^* (\hat{v})).
\]

By definition,

\[
x^* (v_i, \check{v}_{-i}) \in \arg \max_{x \in X} \left[ u_i (x, v_i) + \sum_{j \neq i} u_j (x, \hat{v}_j) - C(x) \right],
\]

so \(\check{v}_i = v_i\) is a weakly dominant strategy. Moreover, pick any function \(h_i : \times_{j \neq i} V_j \to R\) and note that if

\[
t_i (\hat{v}) = C(x^* (\hat{v})) - \sum_{j \neq i} u_j (x^* (\hat{v}), \hat{v}_j) + h_i (\check{v}_{-i})
\]

then

\[
\pi_i (\hat{v}) = u_i (x^* (\hat{v}), v_i) + \sum_{j \neq i} u_j (x^* (\hat{v}), \hat{v}_j) - C(x^* (\hat{v})) - \underbrace{h_i (\check{v}_{-i})}_{\text{constant}}.
\]

so \(\check{v}_i = v_i\) is still a weakly dominant strategy.

### 1.2.5 Complete versus Incomplete Information

Notice that the normal form that we have specified assumes that players have complete information about the payoff functions. Usually, the way we think about the Groves mechanism is as a way to truthfully elicit preferences. To make sense of this in a complete information setup we can imagine that a “planner” is uninformed, while the agents know everything about each other. However, when looking at dominance in this particular class of environment it turns out that the analysis carries over to the case with incomplete information. We may discuss this when we study games of incomplete information.
1.2.6 Why it Works: Internalizing the Externality

Suppose that \( i \) wouldn’t exist. Then the efficient choice would be

\[
\max_{x \in X} \sum_{j \neq i} u_j (x, \hat{v}_j) - C(x) = S_i (v_{-i})
\]

Hence, the external effect on the rest of the economy from being added to the economy is

\[
\sum_{j \neq i} u_j (x^* (v), v_j) - C(x^*(v)) - S_i (v_{-i}),
\]

so the interpretation of the Groves transfer is that each agent is being paid the externality it generates on all others (if positive) or needs to pay the externality imposed on all others (if negative).

1.2.7 A Vickrey Auction is a Special Case of the Groves Mechanism

Let

1. \( u_i (x, v_i) = x_i v_i \)
2. \( X = \{ x \in R^n_+ | x_i \geq 0 \text{ for each } i \text{ and } \sum_{i=1}^n x_i = 1 \} \)
3. \( C(x) = 0 \text{ for each } x \in X \)
4. \( V_i = R_+ \)

Then,

\[
x^*(\hat{v}) \in \arg\max_x \sum_{i=1}^n x_i \hat{v}_i
\]

s.t. \( x_i \geq 0 \) for each \( i \)

\[
\sum_{i=1}^n x_i = 1.
\]

Obviously ties can be broken any way, but one solution is that

\[
x_i^*(\hat{v}) = \begin{cases} 
\frac{1}{|\arg\max_{j \in N} \hat{v}_j|} & \text{if } \hat{v}_i = \max_j \hat{v}_j \\
0 & \text{if } \hat{v}_i < \max_j \hat{v}_j
\end{cases}
\]
Now
\[ t_i(\hat{v}) = -\sum_{j\neq i} u_j(x^*(\hat{v}), \hat{v}_j) = \begin{cases} 0 & \text{if } \hat{v}_i > \max_{j\neq i} \hat{v}_j \\ \left(1 - \frac{1}{|\arg\max_{j\in N} \hat{v}_j|}\right) \max_{j\neq i} \hat{v}_j & \text{if } \hat{v}_i = \max_{j\neq i} \hat{v}_j \\ -\max_{j\neq i} \hat{v}_j & \text{if } \hat{v}_i < \max_{j\neq i} \hat{v}_j \end{cases} \]

Hence, the Groves mechanism boils down to an “auction” where the winner doesn’t pay and the loser is paid the announced valuation of the winner. However, just add \( h_i(\hat{v}_{-i}) = \max_{j\neq i} \hat{v}_j \) to the transfer and we get
\[ t_i(\hat{v}) = \begin{cases} \max_{j\neq i} \hat{v}_j & \text{if } \hat{v}_i > \max_{j\neq i} \hat{v}_j \\ \frac{1}{|\arg\max_{j\in N} \hat{v}_j|} \max_{j\neq i} \hat{v}_j & \text{if } \hat{v}_i = \max_{j\neq i} \hat{v}_j \\ 0 & \text{if } \hat{v}_i < \max_{j\neq i} \hat{v}_j \end{cases} \]

We can thus conclude that the second price auction is a special case of a Groves mechanism for an environment where a single indivisible object is to be allocated to an agent.

### 1.2.8 Groves Mechanisms for a Binary Public Good

Now, assume instead that

1. \( u_i(x, v_i) = xv_i \)
2. \( X = [0, 1] \)
3. \( C(x) = xc \) for each \( x \in [0, 1] \)
4. \( V_i = R_+ \)

Now
\[ x_i^*(\hat{v}) \in \arg\max_{x\in[0,1]} x \left(\sum_{i=1}^n \hat{v}_i - c\right) \]

or (ignoring the indifference)
\[ x_i^*(\hat{v}) = \begin{cases} 1 & \text{if } \sum_{i=1}^n \hat{v}_i - c > 0 \\ 0 & \text{if } \sum_{i=1}^n \hat{v}_i - c < 0 \end{cases} \]
so that

\[ t_i(\tilde{v}) = \begin{cases} 
  c - \sum_{j \neq i} \tilde{v}_i & \text{if } \sum_{i=1}^{n} \tilde{v}_i - c > 0 \\
  0 & \text{if } \sum_{i=1}^{n} \tilde{v}_i - c < 0 
\end{cases}, \]

which is often referred to as a “pivot mechanism”.

1.3 Iterated Elimination of Strictly Dominated Strategies

The reader should be warned that the standard approach to dominance as well as iterations of dominance relies on randomized strategies. We will deal with that later.

However, if a strategy is strictly dominated it is not a best response to anything the opponent(s) can do. Rationality would therefore rule out play of strictly dominated strategies. But, assuming that all agents understand the structure of the game, then other agents should understand that strictly dominated strategies will not be played, so one should be able to eliminate these. Hence, we can ask again whether there are additional strategies that are strictly dominated given the smaller set of surviving strategies.

1.3.1 A Simple Example

Consider

\[
\begin{array}{ccc}
L & C & R \\
U & 1,0 & 1,2 & 0,1 \\
D & 0,3 & 0,1 & 2,0 \\
\end{array}
\]

and note:

1. R is strictly dominated by C. Hence, player 1 (the row player) should make the inference that R will not be played and consider the reduced game

\[
\begin{array}{cc}
L & C \\
U & 1,0 & 1,2 \\
D & 0,3 & 0,1 \\
\end{array}
\]
2. In the reduced game D is strictly dominated by U, so the “game” that remains is

\[
\begin{array}{c|cc}
L & C \\
\hline
U & 1, 0 & 1, 2 \\
\end{array}
\]

Player 2 should then reason that player 1 will predict that he will not play the strictly dominated strategy R and that 1 consequently will not play D.

3. Hence, player 2 should pick C and (U,C) is the unique strategy profile that survives iterated elimination of strictly dominated strategies.

### 1.3.2 A Linear Cournot Duopoly

Suppose that inverse demand is \( P(Q) = \max \{2 - Q, 0\} \) and that two firms produce a homogenous good at a constant unit cost \( c = 1 \). Assuming firms maximize profits their payoff functions are then

\[
\pi_1(q_1, q_2) = \max \{q_1 (1 - q_1 - q_2), -q_1\}
\]

\[
\pi_1(q_1, q_2) = \max \{q_1 (1 - q_1 - q_2), -q_1\}
\]

which immediately allows us to conclude that any \( q_i > 1 \) is strictly dominated. Hence, the problem for \( i \) simplifies to

\[
\max_{q_i} q_i (1 - q_i - q_j)
\]

which has solution \( q_i = \frac{1 - q_j}{2} \), which is strictly decreasing in \( q_j \). We therefore conclude that any \( q_i > \frac{1}{2} \) is strictly dominated as then

\[
q_i (1 - q_i - q_j) - \frac{1}{2} \left(1 - \frac{1}{2} - q_j\right) = q_i (1 - q_i) - \frac{1}{2} \left(1 - \frac{1}{2}\right) - \left(q_i - \frac{1}{2}\right) q_j
\]

\[
< q_i (1 - q_i) - \frac{1}{2} \left(1 - \frac{1}{2}\right) < 0
\]

where the final inequality holds because \( \frac{1}{2} \) solves the monopoly problem \( \max_q q (1 - q) \). But then we may conclude that \( q_i < \frac{1}{4} \) is strictly dominated in the reduced game where quantities
are picked in \([0, \frac{1}{2}]\) as \(q_i < \frac{1}{4}\) implies that

\[
q_i (1 - q_i - q_j) - \frac{1}{4} \left(1 - \frac{1}{4} - q_j\right) = q_i \left(1 - q_i - \frac{1}{2}\right) - \frac{1}{4} \left(1 - \frac{1}{4} - \frac{1}{2}\right) + \left(q_i - \frac{1}{4}\right) \left(\frac{1}{2} - q_j\right).
\]

\[
\leq q_i \left(1 - q_i - \frac{1}{2}\right) - \frac{1}{4} \left(1 - \frac{1}{4} - \frac{1}{2}\right) < 0,
\]

where the last inequality holds because \(\frac{1}{4}\) maximizes \(q_i \left(1 - q_i - \frac{1}{2}\right)\).

In general, assume that after \(n\) recursions we are left with \([a_n, b_n]\). Then, after \(n + 1\) recursions we are left with \([a_{n+1}, b_{n+1}]\) where

\[
a_{n+1} = \frac{1 - b_n}{2},
\]

\[
b_{n+1} = \frac{1 - a_n}{2}.
\]

This can be seen by observing that if \(q_i < \frac{1 - b_n}{2}\) then,

\[
q_i (1 - q_i - q_j) - \frac{1 - b_n}{2} \left(1 - \frac{1 - b_n}{2} - q_j\right) = q_i (1 - q_i - b_n) - \frac{1 - b_n}{2} \left(1 - \frac{1 - b_n}{2} - b_n\right) + \left(q_i - \frac{1 - b_n}{2}\right) (b_n - q_j).
\]

\[
\leq q_i (1 - q_i - b_n) - \frac{1 - b_n}{2} \left(1 - \frac{1 - b_n}{2} - b_n\right) < 0
\]

as \(\frac{1 - b_n}{2}\) maximizes \(q_i (1 - q_i - b_n)\). Moreover, if \(q_i > \frac{1 - a_n}{2}\) then

\[
q_i (1 - q_i - q_j) - \frac{1 - a_n}{2} \left(1 - \frac{1 - a_n}{2} - q_j\right) = q_i (1 - q_i - a_n) - \frac{1 - a_n}{2} \left(1 - \frac{1 - a_n}{2} - a_n\right) + \left(q_i - \frac{1 - a_n}{2}\right) (a_n - q_j).
\]

\[
\leq q_i (1 - q_i - a_n) - \frac{1 - a_n}{2} \left(1 - \frac{1 - a_n}{2} - a_n\right) < 0
\]

as \(\frac{1 - a_n}{2}\) maximizes \(q_i (1 - q_i - a_n)\).

Hence, the set of strategies that survive iterated elimination of strictly dominated strategies is \(q_i \in [a_n, b_n]\) for every \(n\). But

\[
a_{n+1} = \frac{1 - \frac{1 - a_n}{2}}{2} = \frac{1 + a_{n-1}}{4},
\]

\[
b_{n+1} = \frac{1 - \frac{1 - a_n}{2}}{2} = \frac{1 + b_{n-1}}{4},
\]
so:

- \( a_{n+1} > a_{n-1} \) provided that \( \frac{1+a_{n-1}}{4} > a_{n-1} \Leftrightarrow a_{n-1} < \frac{1}{3} \) and \( a_{n+1} < \frac{1}{3} \) provided that \( \frac{1+a_{n-1}}{4} < \frac{1}{3} \Leftrightarrow a_{n-1} < \frac{1}{3} \). Hence, since \( a_1 = 0 \) it follows by induction that \( a_n \in (0, \frac{1}{3}) \) for each \( n \), implying that \( \{a_n\}_{n=1}^{\infty} \) is an increasing sequence.

- \( b_{n+1} < b_{n-1} \) provided that \( \frac{1+b_{n-1}}{4} < b_{n-1} \Leftrightarrow b_{n-1} > \frac{1}{3} \) and \( b_{n+1} > \frac{1}{3} \) provided that \( \frac{1+b_{n-1}}{4} > \frac{1}{3} \Leftrightarrow b_{n-1} > \frac{1}{3} \). Hence, since \( b_1 = \frac{1}{2} \) it follows by induction that \( a_n \in (\frac{1}{3}, \frac{1}{2}) \) for each \( n \), implying that \( \{b_n\}_{n=1}^{\infty} \) is a decreasing sequence.

- \( \{a_n\}_{n=1}^{\infty} \) and \( \{b_n\}_{n=1}^{\infty} \) are taking on values in \([0,1]\). Monotone convergence theorem implies existence of limits:

Hence \( a_n \to a^* \) and \( b_n \to b^* \) where

\[
\begin{align*}
a^* &= \frac{1 + a^*}{4} \\
b^* &= \frac{1 + b^*}{4}
\end{align*}
\]

or \( a^* = b^* = \frac{1}{3} \).

### 1.4 Iterated Elimination of Weakly Dominated Strategies

Consider

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>U</td>
<td>1,0</td>
<td>0,1</td>
</tr>
<tr>
<td>D</td>
<td>0,0</td>
<td>0,2</td>
</tr>
</tbody>
</table>

- L is weakly (strictly too) dominated. Once it is eliminated nothing more can be eliminated. Hence, the prediction is (U,R) and (D,R).

- D is weakly dominated. Once it is eliminated L can be eliminated. Hence (U,R) is all that remains.

- Hence, the order matters.
2 Nash Equilibrium

John Nash suggested that a plausible way to define equilibria in games is to postulate:

1. Every player does his or her best against what other players are doing (utility maximization), and;

2. All players have rational expectations.

This leads to the following definition:

**Definition 6** $s^* \in S$ is a (pure strategy) Nash equilibrium if $u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*)$ for each $i$ and every $s_i$ in $S_i$.

2.1 Justifications

As we will see in examples, sometimes the Nash equilibrium concept is a bit problematic as it imposes lots of coordination. However, there are several types of justifications:

- If the players agree in advance to play (a particular) Nash equilibrium, nobody has an incentive to deviate.

- Mass action/Social norms. If a strategic situation occurs frequently, a Nash equilibrium may emerge as a “norm” for how to play.

- Predicting non Nash equilibrium play implies that the game theorist understands the game better than the players being modelled.

- Evolutionary stability. Close connection between Nash equilibria and evolutionary equilibrium concepts.

2.2 The Best Response Correspondance

In simple games we can use the definition of Nash directly and check for equilibria by guess and verify. However, when it gets more complicated we need to be more systematic.
**Definition 7** Given a normal form game \( G = (N, S, u) \) the best response correspondence for player \( i \) is a mapping \( \beta_i : S_{-i} \rightarrow S_i \) defined by

\[
\beta_i (s_{-i}) = \arg \max_{s_i} u_i (s_i, s_{-i}) \\
= \{ s_i \in S_i \text{ such that } u_i (s_i, s_{-i}) \geq u_i (s'_i, s_{-i}) \text{ for all } s'_i \in S_i \}
\]

**Definition 8** Given a normal form game \( G = (N, S, u) \) the best response correspondence is a mapping \( \beta : S \rightarrow S \) where for every \( s \in S \) we have that \( \beta (s) = (\beta_1 (s_{-1}), ..., \beta_n (s_{-n})) \).

Notice that \( \beta_i (s_{-i}) \) in general may be a set as illustrated in the simple example below,

<table>
<thead>
<tr>
<th></th>
<th>Bob</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Left</td>
</tr>
<tr>
<td>Alice</td>
<td>Up</td>
</tr>
<tr>
<td></td>
<td>Down</td>
</tr>
</tbody>
</table>

where \( \beta_1 (\text{Left}) = \{ \text{Up, Down} \} \).

It is pretty clear that:

**Proposition 2** \( s^* \) is a (pure strategy) Nash equilibrium if and only if \( s^* \in \beta (s^*) \) for every player \( i \).

**Proof.**

Suppose that \( s^* \) is a (pure strategy) Nash equilibrium but that \( s^* \) is not in \( \beta (s^*) \). Then there at least one player such that \( s^*_i \notin \beta_i (s^*_{-i}) \) implying that there exists \( s'_i \) such that \( u_i (s'_i, s^*_{-i}) > u_i (s^*) \). Hence, there is a contradiction as this says that \( s^* \) is not a Nash equilibrium for every player \( i \)

Next, if \( s^* \in \beta (s^*) \) then \( s^*_i \in \beta_i (s^*_{-i}) \) for each player so that

\[
s^*_i \in \arg \max_{s_i \in S_i} u_i (s_i, s^*_{-i}) ,
\]

for each player \( i \) which means that \( s^* \) is a Nash equilibrium. \( \blacksquare \)
2.3 Finding Nash Equilibria By Using the Best Response Correspondance

2.3.1 The Algorithm

Consider the Game

\[
\begin{array}{ccc}
\text{Bob} & x & y & z \\
\text{Alice} & a & 1,1 & 2,0 & 3,-1 \\
& b & -1,1 & 6,1 & 8,7 \\
& c & 3,0 & -10,0 & 6,-5 \\
& d & 3,2 & 0,0 & -5,1 \\
\end{array}
\]

which doesn’t have any particular interpretation. Begin by putting in the best responses for Alice by underlining the corresponding payoff in the payoff bi-matrix as follows

\[
\begin{array}{ccc}
\text{Bob} & x & y & z \\
\text{Alice} & a & 1,1 & 2,0 & 3,-1 \\
& b & -1,1 & 6,1 & 7,7 \\
& c & 3,0 & -10,0 & 6,-5 \\
& d & 3,2 & 0,0 & -5,1 \\
\end{array}
\]

Do the same for Bob

\[
\begin{array}{ccc}
\text{Bob} & x & y & z \\
\text{Alice} & a & 1,1 & 2,0 & 3,-1 \\
& b & -1,1 & 6,1 & 7,7 \\
& c & 3,0 & -10,0 & 6,-5 \\
& d & 3,2 & 0,0 & -5,1 \\
\end{array}
\]

We conclude that \((c, x)\), \((d, x)\) and \((b, x)\) are Nash equilibria.
2.3.2 Cournot Duopoly with Linear Demand and Constant Unit Cost Zero

Consider the Cournot duopoly model with inverse demand \( P(q) = 2 - q \) and \( C_i(q_i) = c q_i \). Then (ignoring \((q_1, q_2)\) such that \( q_1 + q_2 > 2 \), which cannot be the case in an equilibrium).

\[
\pi_1(q_1, q_2) = q_1 \left(1 - q_1 - q_2\right)
\]
\[
\pi_1(q_1, q_2) = q_2 \left(1 - q_1 - q_2\right)
\]

We will solve for a Nash equilibrium by using the best response. Hence, \( B_1(q_2) \) is found by solving

\[
\beta_1(q_2) = \arg\max_{q_1} q_1 \left(1 - q_1 - q_2\right) = \frac{1 - q_2}{2}
\]
\[
\beta_2(q_1) = \arg\max_{q_2} q_2 \left(1 - q_1 - q_2\right) = \frac{1 - q_1}{2}.
\]

In a Nash equilibrium

\[
q_1^* = \beta_1(q_2^*) = \frac{1 - q_2^*}{2}
\]
\[
q_2^* = \beta_2(q_1^*) = \frac{1 - q_1^*}{2}.
\]

Solving we obtain \((q_1^*, q_2^*) = \left(\frac{1}{3}, \frac{1}{3}\right)\), which is the unique Nash equilibrium of the game. We notice that this is also the solution by iterated elimination of strictly dominated strategies. This is a general fact:

**Proposition 3** Suppose that \( s^* \) is the only strategy profile that survives iterated elimination is strictly dominated strategies. Then, \( s^* \) is a Nash equilibrium.

**Proof.** Let \( \{S^k\}_{k=1}^{\infty} \) be a sequence of subsets of \( S \) where \( S^k \) is the set of strategies that survive \( k \) rounds of elimination (if the game is finite, the process will stop after a finite number of steps, but this is irrelevant for the argument). If \( s^* \) survives the process of elimination it means that for each \( k \) and every \( s_i \in S^k_i \) there exists \( s_{-i} \in S^k_{-i} \) such that

\[
u_i(s_i^*, s_{-i}) \geq u_i(s_i, s_{-i})
\]
holds. Assume that \( s^* \) is not a Nash equilibrium. Then, there exists \( i \in N \) and \( s_i \in S_i \) such that

\[
    u_i \left( s_i, s^*_{-i} \right) > u_i \left( s^*_i, s^*_{-i} \right),
\]

which implies that \( (s_i, s^*_{-i}) \) survives the process of elimination because \( s^*_{-i} \in S_{-i}^k \) for every \( k \). This contradicts the assumption that \( s^* \) is the unique profile surviving iterated elimination of strictly dominated strategies.

Also,

**Proposition 4** Suppose that \( s^* \) is a Nash equilibrium, then \( s^* \) survives iterated elimination of strictly dominated strategies.

**Proof.** Again, let \( \{S^k\}_{k=1}^\infty \) be a sequence of subsets of \( S \) where \( S^k \) is the set of strategies that survive \( k \) rounds of elimination. For contradiction, suppose that \( s^* \) does not survive iterated elimination of strictly dominated strategies. Then, let \( k^* \) be the first round where there exist \( i \in N \) such that \( s^*_i \) is eliminated, implying that

\[
    u_i \left( s^*_i, s_{-i} \right) < u_i \left( s_i, s_{-i} \right)
\]

holds for every \( s_{-i} \in S_{-i}^{k^*-1} \). But, by construction, \( s^*_i \in S_{-i}^{k^*-1} \), so

\[
    u_i \left( s^*_i, s^*_{-i} \right) < u_i \left( s_i, s^*_{-i} \right),
\]

contradicting the assumption that \( s^* \) is a Nash equilibrium.